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Boundary Conditions for Viscous Vortex Methods

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This paper presents a Neumann-type vorticity boundary condition for the vorticity formulation of the Navier–Stokes equations. The vorticity creation process at the boundary, due to the no-slip condition, is expressed in terms of a vorticity flux. The scheme is incorporated then into a Lagrangian vortex blob method that uses a particle strength exchange algorithm for viscous diffusion. The no-slip condition is not enforced by the generation of new vortices at the boundary but instead by modifying the strength of the vortices in the vicinity of the boundary. © 1994 Academic Press, Inc.

INTRODUCTION

In this paper we present a scheme that concerns the enforcement of the no-slip boundary condition in the vorticity-velocity formulation of the unsteady Navier–Stokes equations. We are interested in the formulation of the problem and its application to a Lagrangian vortex blob method. A fractional step algorithm is employed and the vorticity creation process is modelled by a vorticity flux on the surface of the body. This flux is appropriately distributed to the computational elements, thus modifying their vorticity so that the no-slip boundary condition is enforced.

In the vorticity-velocity formulation of the boundary conditions we may distinguish schemes that involve the boundary value of the vorticity [19] or the vorticity flux (Kinney and coworkers [10, 11]). In the context of vortex method Chorin [3] introduced the idea of creating vortex blobs at the surface of the body in order to enforce the no-slip boundary condition. The existence of such blobs on the boundary, though, introduced a smoothing region for the vorticity on the boundary that significantly increases the numerical diffusion of the scheme. To alleviate this difficulty Chorin [4] proposed the vortex sheet method based on a coupling of the solution of the Prandtl boundary layer equations near the body and the full Navier–Stokes equations away from it. However, this approach seems to encounter difficulties with fully separated flows [1], where a well-defined boundary layer region fails to exist, and introduces many parameters as to the region of validity of the boundary layer equations, the transformation of vortex sheets into vortex blobs, etc. Moreover, according to the sheet model the number of computational elements increases at each time step as new elements are introduced in the fluid in order to satisfy the no-slip condition.

In the present method we implement an alternative technique to enforce the no-slip boundary condition in the context of vortex methods. A Dirichlet-type condition would explicitly require the value of the vorticity at the wall. The computation of such a quantity is prone to interpolation errors that are further augmented by the use of a Lagrangian grid. On the other hand, the present formulation of a Neumann-type condition is not prone to such interpolation errors and does not require additional computational elements (vortex blobs) on the surface of the body. Based on the implementation presented herein this vorticity flux is distributed by diffusion to the existing blobs, thus altering their strength but without increasing their population.

In the context of vortex methods this consists of modifying the strength of the existing particles in the vicinity of the boundary. This algorithm presents, then, an alternative scheme to Chorin's algorithm of particle creation. It eliminates some of the arbitariness of the above-mentioned scheme and provides accurate results without the need for a special type of elements (vortex sheets). It complements and extends the scheme of particle strength exchange (for the unbounded domain [6] and for the bounded one-dimensional case [16] that accounts for diffusion, as all viscous effects are treated by modifying the strength of the vortices.

1. THE 2D NAVIER-STOKES EQUATIONS

Two-dimensional incompressible unsteady flow of a viscous fluid may be described in terms of the velocity ($\mathbf{u}(\mathbf{x}, t)$) and the vorticity ($\nabla \times \mathbf{u} = \boldsymbol{\omega} = \omega \hat{\mathbf{e}}_z$) of the flow as

$$\frac{\partial \omega}{\partial t} + \mathbf{u} \cdot \nabla \omega = \nu \nabla^2 \omega. \tag{1}$$

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Here ν denotes the kinematic viscosity of the fluid. When the flow is around a solid configuration (translating with velocity $\mathbf{U}_b(t)$ and rotating around its center of mass ($\mathbf{x}_b$) with angular velocity $\Omega(t)$), boundary conditions need to be enforced. On the surface of the body ($\mathbf{x}_s$) the velocity of the fluid ($\mathbf{u}$) is equal to the velocity of the body ($\mathbf{U}_s$):

$$\mathbf{u}(\mathbf{x}_s) = \mathbf{U}_s \tag{1a}$$

with

$$\mathbf{U}_s = \mathbf{U}_b(t) + \Omega(t)\,\hat{\mathbf{e}}_z \times (\mathbf{x}_s - \mathbf{x}_b)$$

and at infinity,

$$\mathbf{u}(\mathbf{x}) \to \mathbf{U}_\infty \qquad \text{as} \quad |\mathbf{x}| \to \infty, \tag{1b}$$

where $\mathbf{U}_\infty$ is the free stream velocity. Using the definition of the vorticity and the continuity ($\nabla \cdot \mathbf{u} = 0$) it can be shown that $\mathbf{u}$ is related to $\boldsymbol{\omega}$ by the Poisson equation,

$$\nabla^2 \mathbf{u} = -\nabla \times \boldsymbol{\omega}. \tag{2}$$

The velocity-vorticity formulation helps in eliminating the pressure from the unknowns of the equations. However, for bounded domains it introduces additional constraints in the kinematics of the flow field and requires the transformation of the velocity boundary conditions to vorticity form. The proposed numerical method is based on the discretization of the above equations in a Lagrangian frame using particle (vortex) methods.

1.1. *Particle (Vortex) Methods*

The vorticity equation, Eq. (1), may be expressed in a Lagrangian formulation by solving for the vorticity carrying fluid elements ($\mathbf{x}_a$) based on the following set of equations:

$$\begin{aligned} \frac{d\mathbf{x}_a}{dt} &= \mathbf{u}(\mathbf{x}_a, t) \\ \frac{d\omega}{dt} &= \nu \nabla^2 \omega. \end{aligned} \tag{3}$$

In the context of particle methods it is desirable to replace the right-hand side of Eqs. (3) by integral operators. These operators are discretized using the locations of the particles as quadrature points so that ultimately Eqs. (3) are replaced by a set of ODEs whose solution is equivalent to the solution of the original set of equations.

To this effect the velocity field may be determined by the vorticity field using the Green's function formulation for the solution of Poisson's equation (Eq. (2)),

$$\mathbf{u} = -\frac{1}{2\pi} \int \mathbf{K}(\mathbf{x} - \mathbf{y}) \times \boldsymbol{\omega}\, d\mathbf{y} + \mathbf{U}_0(\mathbf{x}, t),$$

where $\mathbf{U}_0(\mathbf{x}, t)$ contains the contribution from the solid body rotation and $\mathbf{U}_\infty$, and $\mathbf{K}(\mathbf{z}) = \mathbf{z}/|\mathbf{z}|^2$. The use of the Biot–Savart law to compute the velocity field guarantees the enforcement of the boundary condition at infinity.

The Laplacian operator may be approximated by an integral operator [6] as well so that

$$\nabla^2 \omega \approx \int G_\varepsilon(|\mathbf{x} - \mathbf{y}|)[\omega(\mathbf{x}) - \omega(\mathbf{y})]\, d\mathbf{y},$$

where, in this paper, G_ε is taken to be $G_\varepsilon(\mathbf{z}) = (2/\pi\varepsilon^2)\exp(-|\mathbf{z}|^2/2\varepsilon^2)$. The boundary condition Eq. (1a) is enforced by formulating the physical mechanism it describes. The solid wall is the source of vorticity that enters the flow. A vorticity flux ($\partial\omega/\partial n$) may be determined on the boundary in a way that ensures that Eq. (1a) is satisfied. Here a fractional step algorithm is presented that allows for the calculation of this vorticity flux. It is shown then that this mechanism of vorticity generation can be expressed by an integral operator as well,

$$\frac{d\omega}{dt} = \int H(\mathbf{x}, \mathbf{y}) \frac{\partial\omega}{\partial n}(\mathbf{y})\, d\mathbf{y},$$

where the kernel H is developed in Section 3. Using the above integral representations for the right-hand side of Eq. (3) we obtain the following set of equations:

$$\begin{aligned} \frac{d\mathbf{x}_a}{dt} &= -\frac{1}{2\pi} \int \mathbf{K}(\mathbf{x}_a - \mathbf{y}) \times \boldsymbol{\omega}\, d\mathbf{y} + \mathbf{U}_0(\mathbf{x}_a, t) \\ \frac{d\omega}{dt} &\approx \nu \int G_\varepsilon(|\mathbf{x}_a - \mathbf{y}|)[\omega(\mathbf{x}_a) - \omega(\mathbf{y})]\, d\mathbf{y} \\ \frac{d\omega}{dt} &\approx \nu \int H(\mathbf{x}_a, \mathbf{y}) \frac{\partial\omega}{\partial n}(\mathbf{y})\, d\mathbf{y}. \end{aligned} \tag{4}$$

In vortex methods, the vorticity field is considered as a discrete sum of the individual vorticity fields of the particles, having core radius ε, strength $\Gamma(t)$, and an individual distribution of vorticity determined by the function η_ε so that

$$\omega(\mathbf{x}, t) = \sum_{n=1}^{N} \Gamma_n(t)\, \eta_\varepsilon(\mathbf{x} - \mathbf{x}_n(t)).$$

When this expression for the vorticity is substituted in Eq. (4) the singular integral operators K, G are convolved with the smooth function η_ε and are replaced by smooth operators K_ε, G_ε. These integrals are subsequently discretized using a quadrature having as quadrature points the locations of the particles. Assuming that each particle occupies a region of area h^2 and that the shape of the body is discretized by M panels then algorithmically the method may be expressed as:

$$\begin{aligned} \frac{d\mathbf{x}_i}{dt} &= -\frac{1}{2\pi}\sum_{j=1}^{N} \Gamma_j K_\varepsilon(\mathbf{x}_i - \mathbf{x}_j) + \mathbf{U}_0(\mathbf{x}_i, t) \\ \frac{d\Gamma_i}{dt} &= \nu \sum_{j=1}^{N} [\Gamma_j - \Gamma_i]\, G_\varepsilon(|\mathbf{x}_i - \mathbf{x}_j|) \\ \frac{d\Gamma_i}{dt} &= \nu \sum_{m=1}^{M} H(\mathbf{y}_i, \mathbf{y}_m) \frac{\partial\omega}{\partial n}(\mathbf{y}_m) \\ \Gamma_i(0) &= \omega(\mathbf{x}_i, 0)\, h^2, \qquad i = 1, 2, ..., N. \end{aligned} \tag{5}$$

The characteristic of the present method is the replacement of the differential operators by integral operators. The advantage in employing integral operators is based on their stability and efficiency. Integral operators are bounded and smoothing, so that discrete approximations have a bounded condition number as the mesh is refined. The Lagrangian representation of the convective terms avoids many difficulties associated with its discretization on an Eulerian mesh such as excess numerical diffusion. However, the accuracy of the method relies on the accuracy of the quadrature rule as information needs to be gathered from the possibly scattered particle positions.

2. VORTICITY CREATION AT A SOLID WALL—LIGHTHILL'S MODEL

The basis of the present formulation was originally proposed by Lighthill [15]. The key observation is that once the vorticity field is known then the entire flow field may be determined (via the Biot–Savart law of velocity induction). The vorticity field is convected and diffused in the fluid but in the presence of solid boundaries one has to account for the vorticity production on the solid walls as well. Lighthill models this vorticity creation process by considering the body surface as a collection of vorticity sources and sinks. To calculate these (unknown) vorticity strengths he proposes that the velocity field must be computed on the solid boundary from the known vorticity field. From kinematic considerations, in order to ensure the no-through flow boundary condition, a potential flow correction needs to be superimposed to this velocity field. The resulting velocity field would have (in general) a nonzero tangential velocity component and one may view the surface of the body as a vortex sheet. To ensure the no-slip condition and model the vorticity creation process on the solid boundary this vortex sheet has to be related to the vorticity production at the wall. Lighthill concludes the description of his model by stating that the vorticity per unit area has been created and is equal to the negative of this vortex sheet strength. What remains incomplete in this model is how this vorticity enters the fluid adjacent to the wall or how the vortex sheet strength may be incorporated in a vorticity-type boundary condition.

One may observe that the strength of the vortex sheet has dimensions of velocity (or length over time). To obtain an appropriate (dimensionally correct) vorticity boundary condition this vortex sheet strength can be manipulated so that a Dirichlet-type (vorticity with dimensions 1/time) or Neumann-type (vorticity flux with dimensions of acceleration) may be obtained. This is basically the point of diversion of the various formulations involving vorticity boundary conditions. Chorin [3] divides the strength of the vortex sheet by a length equal to the elementary discretization length on the body surface, whereas Wu [19] divides it by the distance from the wall to the first mesh point in the computational domain, to obtain the vorticity on the body. Kinney and his coworkers [10, 11] envision this vortex sheet as equivalent to a vorticity flux over a small time interval (thus dividing the sheet strength by time to obtain the proper units of acceration). An integral constraint is imposed on all formulations on the vorticity created at the wall so as to satisfy Kelvin's theorem of production of circulation.

In the present work the Neumann type vorticity boundary condition was chosen. This choice was mainly dictated by the use of vortex methods for the resolution of the vorticity transport equation and it meshes well with the scheme of particle strength exchange used for diffusion.

2.1. *Mathematical Formulation*

We consider a body translating with velocity $\mathbf{U}_b(t)$ and rotating with angular velocity $(\Omega(t))$ immersed in a flow field induced by a uniform flow $(\mathbf{U}_\infty)$ and vorticity in the wake $(\omega_f(\mathbf{x}, t))$. In the present formulation the solid body is represented by suitable vorticity distributions determined so that the entire flow field follows the prescribed solid body motion. The interior of the body is replaced by a uniform vorticity field (ω_b) with strength equal to twice the magnitude of the angular velocity $(\omega_b = 2\Omega)$, while the surface of the body is replaced by a vortex sheet (bound vorticity) with strength $\gamma(s)$ (Fig. 1).

As discussed by Lamb [14], the kinematic velocity field is uniquely determined from the vorticity field if the no-through flow boundary condition $((\mathbf{u} - \mathbf{U}_s) \cdot \mathbf{n} = 0)$ has been enforced on the surface of the body. A vortex sheet appears then on the surface of the body and the enforcement of the no-through flow boundary condition is equivalent to determining the strength of this vortex sheet. The strength of the vortex sheet is computed using the streamfunction of the flowfield. The resulting integral equation is given by

$$\gamma(s) - \frac{1}{\pi}\oint \frac{\partial}{\partial n}[\operatorname{Log}|\mathbf{x}(s) - \mathbf{x}(s')|]\,\gamma(s')\,ds' = -2h(\mathbf{x}(s)), \tag{6}$$

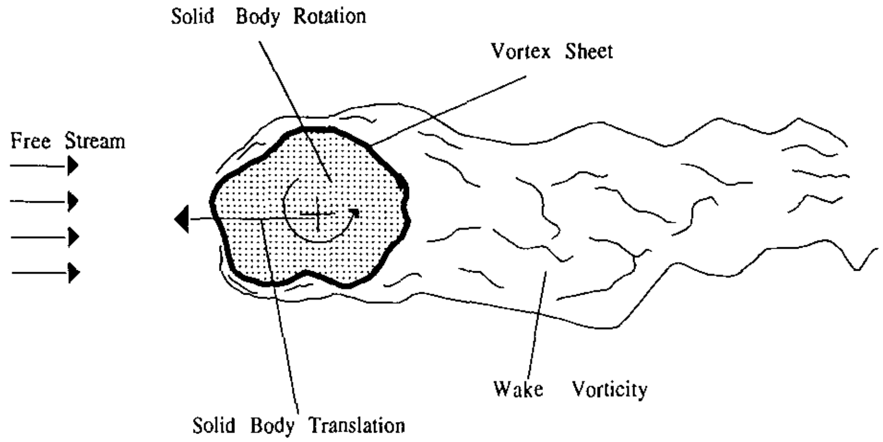

FIG. 1. Sketch showing the different contributions to the flow field of a body in translation and rotation immersed in a viscous incompressible flow field.

where

$$h(\mathbf{x}(s)) = \frac{\partial \Psi_f}{\partial n}(\mathbf{x}(s)) + \frac{\partial \Psi_\Omega}{\partial n}(\mathbf{x}(s)) - \mathbf{U}_s \cdot \mathbf{s}. \quad (7)$$

The solution properties of the above equation were originally studied by Prager [18] who first introduced the concept of replacing the body surface by a continuous vortex sheet. Equation (6) is singular, as it admits a non-unique solution and requires that an extra constraint be imposed on the strength of the vortex sheet. This property of the equation is a fortuitous result, however, as it allows for the coupling of the kinematic description of the flow field with the viscous wall production of vorticity.

The model presented herein relies on the nullification of this spurious vortex sheet at the body surface so as to enforce the no-slip boundary condition. As this vortex sheet is a constituent of the flow field its strength should account for the modification of the circulation of the flow field. Hence when it is eliminated from the body surface in the interval $[t, t+\delta t]$ the circulation (Γ) of the flow field would be modified according to

$$\oint \gamma(s)\, ds = \int_t^{t+\delta t} \frac{d\Gamma}{dt'}\, dt'. \quad (8)$$

On the other hand, Kelvin's theorem states that the rate of change of circulation in the flow field is defined as

$$\frac{d\Gamma}{dt} = \nu \oint \frac{\partial \omega}{\partial n}(s)\, ds = -2\frac{d\Omega}{dt} A_B, \quad (9)$$

where A_B is the area of the body. Integrating Eq. (9) in the interval $[t, t+\delta t]$ we obtain that

$$\int_t^{t+\delta t} \frac{d\Gamma}{dt'}\, dt' = \int_t^{t+\delta t} dt' \oint \nu \frac{\partial \omega}{\partial n}(s)\, ds$$
$$= -2A_B[\Omega(t+\delta t) - \Omega(t)]. \quad (10)$$

Comparing Eq. (8) and Eq. (10) the strength of the vortex sheet may be related to the vorticity flux at the body surface as

$$\nu \int_t^{t+\delta t} \frac{\partial \omega}{\partial n}(s)\, dt' = -\gamma(s) \quad (11)$$

or, if we consider this vorticity flux to be constant over the small interval of time (δt),

$$\nu \frac{\partial \omega}{\partial n}(s) = -\gamma(s)/\delta t. \quad (12)$$

This constitutes then a Neumann-type vorticity boundary condition equivalent to the no-slip boundary condition as expressed by Eq. (1a). The above formulation allows us then to impose an integral constraint on the strength of the vortex sheet γ,

$$\oint \gamma(s)\, ds = -2A_B[\Omega(t+\delta t) - \Omega(t)], \quad (13)$$

and provides a closure for Eq. (6) that admits now a unique solution.[1] Equation (6) and Eq. (13) may be discretized using a boundary element ("panel") method resulting in a well-conditioned system of equations [13]. A few observations should be made here as to the behavior of the above formulation in the limit of $\delta t \to 0$. First note that at the end of a time step the spurious vortex sheet has been eliminated. The strength of the vortex sheet is dependent on the external flow field and the limit of a vanishing δt, there would be accordingly a vanishingly small change for γ as well, so that the vorticity flux would remain finite. The numerical vor-

[1] Note that in Hydrodynamics the equivalent constraint would be a Kutta type condition.

ticity flux should be consistent with the actual vorticity flux as computed by applying the momentum equation on the wall. In body-fitted coordinates this results in

$$\mathbf{s} \cdot \frac{d\mathbf{u}}{dt}\bigg|_{\text{wall}} = -\frac{1}{\rho}\frac{\partial p}{\partial s}\bigg|_{\text{wall}} + \nu \frac{\partial \omega}{\partial n}\bigg|_{\text{wall}}. \tag{14}$$

Vorticity is transfered to the fluid due to the tangential component of the pressure gradient and a possible acceleration of the body surface. In the present fractional step algorithm this pressure gradient is manifested by a spurious slip velocity observed on the body surface. We may consider this slip velocity as an acceleration "equivalent" to a vorticity flux generated at the wall, so that at each time step Eq. (11) is satisfied.

Once γ has been computed (solving Eq. (13) and Eq. (6) the vorticity flux is determined at the surface of the body according to Eq. (12). This vorticity flux is subsequently distributed to the particles (by appropriately modifying their strength) so that the spurious slip velocity is nullified and the vorticity is generated in the fluid. This technique of enforcing the no-slip boundary condition is consistent with the scheme of particle strength exchange (PSE) that accounts for diffusion. In the present method all viscous effects are resolved by appropriately modifying the strength of the particles.

3. DISTRIBUTION OF THE VORTICITY FLUX

Once the vorticity flux has been computed (via Eq. (12)) it has to be distributed to the particles in the domain so that vorticity enters the fluid. This is achieved in the context of a fractional step algorithm by the solution of a diffusion equation with homogeneous initial conditions and a Neumann boundary condition.

3.1. *Mathematical Formulation*

We consider the diffusion equation for the vorticity $\omega(\mathbf{x}, t)$ with homogeneous initial conditions and boundary conditions of the Neumann type:

$$\begin{aligned} &\omega_t - \nu \nabla^2 \omega = 0 && \text{in } \mathscr{D} \times [0, t] \\ &\omega(\mathbf{x}, 0) = 0 && \text{in } \mathscr{D} \\ &\frac{\partial \omega}{\partial n} = F(\mathbf{x}, t) && \text{on } \partial\mathscr{D} \times [0, t], \end{aligned}$$

where $\mathscr{D}$ denotes the computational domain bounded by the surface of the body ($\partial\mathscr{D}$). The solution of the above equation may be expressed in integral form [9] as

$$\omega(\mathbf{x}, t) = \int_0^t \int_{\partial\mathscr{D}} G(\mathbf{x}, t; \xi, \tau)\, \mu(\xi, \tau)\, ds_\xi\, d\tau, \tag{15}$$

where $\mu(\mathbf{x}, t)$ is determined by the solution of

$$-\frac{1}{2}\mu(\mathbf{x}, t) + \int_0^t \int_{\partial\mathscr{D}} \frac{\partial G}{\partial n}(\mathbf{x}, t; \xi, \tau)\, \mu(\xi, \tau)\, ds_\xi\, d\tau = F(\mathbf{x}, t) \tag{16}$$

with

$$G(\mathbf{x}, t; \xi, \tau) = \frac{1}{4\pi\nu(t-\tau)} \exp\left(-\frac{|\mathbf{x}-\xi|^2}{4\nu(t-\tau)}\right).$$

The resulting expressions for the vorticity field involve integrals only over the surface of the body. Those integrals may be discretized with a boundary integral method by assuming that the surface of the body is composed of a set of discrete panels (straight or curved) and assuming a certain variation (constant, linear, etc.) of the unknown function $\mu(\mathbf{x}, t)$ in space (over the panels) and time.

3.2. *Geometrical Definitions*

In order to explicitly evaluate the integral in Eq. (16) let us consider the geometric representation of the body. From Fig. 2 points on the body are defined by

$$\mathbf{y} = \mathbf{x}_0 + x_p \hat{\mathbf{e}} - \frac{x_p^2}{2\rho}\hat{\mathbf{n}} + \mathscr{O}(x_p^3),$$

where $\rho = 1/\kappa$ is the local radius of curvature of the body. Points in the domain are defined as

$$\mathbf{x} = \mathbf{x}_0 + R\sin(a)\,\hat{\mathbf{e}} + R\cos(a)\,\hat{\mathbf{n}}.$$

Based on the above definitions we have also that

$$d\mathbf{y} = dx_p\,\hat{\mathbf{e}} - \frac{x_p}{\rho}\,dx_p\,\hat{\mathbf{n}}$$

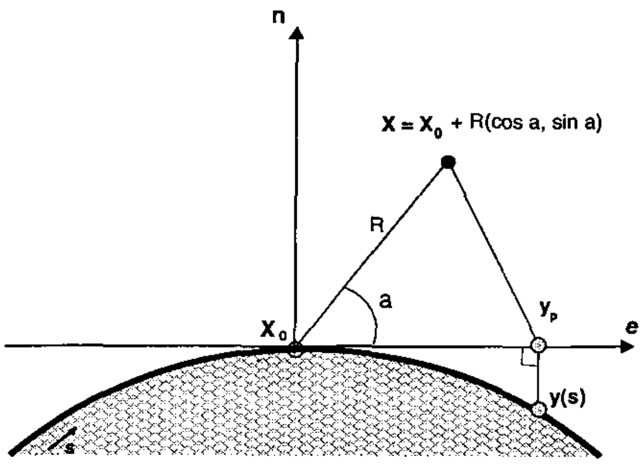

FIG. 2. Definition sketch.

and the distance between the two points is then

$$\|\mathbf{x}-\mathbf{y}\|^2 = R^2 - 2R\sin(a)\,x_p + \left(1 + \frac{R\cos(a)}{\rho}\right)x_p^2 + \mathcal{O}(x_p^3).$$

The evaluation of the integrals in Eq. (15) and Eq. (16) is further simplified if we choose to neglect the terms associated with the local curvature of the body or equivalently describe the body with tangential flat panels. A panel approximation to the body (Fig. 2) then introduces points

$$\mathbf{y_p} = x_p\hat{\mathbf{e}} + \mathbf{x_0}$$

with

$$d\mathbf{y_p} = dx_p\,\hat{\mathbf{e}}$$

and the distance from the field points to the body may be approximated by

$$\|\mathbf{x}-\mathbf{y_p}\|^2 = R^2 - 2R\sin(a)\,x_p + x_p^2.$$

3.3. *Vorticity Evaluation*

Substituting the above approximations for the surface of the body in Eq. (15) we have the following representation for the vorticity field at point $\mathbf{x}$:

$$\omega(R, a) = \int_0^t \frac{1}{4\pi\nu(t-\tau)} \oint_{\partial\mathscr{D}} e^{(-R^2 + 2R\sin(a)x_p - x_p^2)/4\nu(t-\tau)} \times \mu_0(x_p, \tau)\,dx_p\,d\tau.$$

Assuming, furthermore, a constant strength in space, $\mu_i(t)$, for the heat potential over each panel (of size $2d$) we may alternatively express the above equation as

$$\omega = \sum_{i=1}^{M} I_i,$$

where

$$I_i = \int_0^t \mu_i(\tau)\frac{e^{-R^2/4\nu(t-\tau)}}{4\pi\nu(t-\tau)}\,d\tau \int_{-d}^{d} e^{(2R\sin(a)x_p - x_p^2)/4\nu(t-\tau)}\,dx_p.$$

The integral over the panel may then be calculated explicitly so that the vorticity field induced by a panel, i, may be expressed as an integral over time,

$$\omega_i(\mathbf{x}, t) = \tfrac{1}{2}\int_0^t \mu_i(\tau)\,\phi(\mathbf{x}, t-\tau)\,d\tau. \tag{17}$$

with

$$\phi(\mathbf{x}, t-\tau) = \frac{e^{-y^2/4\nu(t-\tau)}}{\sqrt{4\pi\nu(t-\tau)}} \times \left[\operatorname{erf}\left(\frac{d+x}{\sqrt{4\nu(t-\tau)}}\right) + \operatorname{erf}\left(\frac{d-x}{\sqrt{4\nu(t-\tau)}}\right)\right]$$

with $\mathbf{x} = (x, y) = R(\sin a, \cos a)$ (Fig. 2). In order to obtain an explicit expression for the vorticity field we need to assign a time dependence for the potential $\mu(t)$. At first one may wish to assume that the potential remains constant in $[0, t]$. However, we were not able to compute the resulting integrals in a closed form. Alternatively we may wish to compute the integral numerically or assume a δ-function-type dependence of the potential in time, thus enabling us to explicitly evaluate the integral. Hence by setting $\mu_i(t) = \mu_i\delta(t')$ (with t' in $[0, t]$) and assuming for the strength of the particles that $\Gamma_j = \omega(\mathbf{x}_j)\,h_j^2$ we obtain that

$$\Gamma_j(t) = \Gamma_j(0) + \frac{h_j^2}{2}\sum_{i=1}^{M} \mu_i\phi(x_{ij}, y_{ij}, t'). \tag{18}$$

The next section presents an efficient way to compute the values of the surface density μ_i on the panels by a fast evaluation of the double heat potential.

3.4. *Evaluation of the Surface Density*

To complete the evaluation of the vorticity field in the domain the surface density μ is required. Following the derivation of Greengard and Strain [8] Eq. (16) is solved explicitly by exploiting the local character of the Green's function G and its normal derivative on the body. Consider the double layer heat potential which is defined as

$$\mathscr{H}\mu(\mathbf{x}, t) = \oint_{\partial\mathscr{D}} \int_0^t \frac{\partial G}{\partial n}(\mathbf{x}, t; \xi, 0)\,\mu(\xi)\,d\xi\,d\tau.$$

If $\mathbf{s}$ is the coordinate along the boundary of the body, the body may be locally described as

$$x = s, \qquad y = y(s).$$

Using a Taylor's expansion for the shape of the body we have that

$$\begin{aligned} y &= y(0) + y_s(0)s + \frac{1}{2}y_{ss}s^2 = \frac{\kappa}{2}s^2 \\ y_s &= y_s(0) + y_{ss}(0)s = \kappa s \\ \mu &= \mu(0) + \mu_s(0) + \frac{1}{2}\mu_{ss}s^2, \end{aligned} \tag{19}$$

where $\kappa(s)$ is the local curvature of the body. The unit normal to the boundary is given by

$$\mathbf{n} = \frac{[y_s(s), -x_s(s)]}{\sqrt{x_s^2 + y_s^2}} = \frac{[y_s(s), -1]}{\sqrt{1 + y_s^2}}.$$

Differentiating the fundamental solution kernel, it is found that

$$\nabla G = \frac{[-2(x-\xi), -2(y-\zeta)]}{4\nu(t-\tau)} G(\mathbf{x}, t; \xi, \tau).$$

Using the above expansions (Eq. (19)) for the description of the curve and the variation of the surface density along it we obtain

$$\mathscr{H}\mu = 2\int_0^t \oint_{\partial\mathscr{D}} \frac{e^{-(s^2+y^2)/4\nu(t-\tau)}}{4\pi\nu(t-\tau)} \frac{sy_s - y}{4\nu(t-\tau)} \mu(s)\, ds\, d\tau.$$

To evaluate now the above integral we use the transformations

$$z^2 = 4\nu(t-\tau), \qquad s = zr.$$

so that $\mathscr{H}\mu$ becomes (for $t = \delta t$):

$$\mathscr{H}\mu = \frac{1}{\pi\nu}\int_0^{2\sqrt{\nu\delta t}} \int_{-\infty}^{\infty} \left(\frac{y_s}{z} - \frac{y}{z^2}\right) e^{-r^2} e^{-y^2/z^2} \mu(zr)\, dr\, dz.$$

Replacing y and μ by their Taylor expansion yields

$$\mathscr{H}\mu(0, t) \approx \frac{\mu(0)\kappa}{2}\sqrt{\pi\nu\,\delta t} + \mathscr{O}((\nu\,\delta t)^{3/2}).$$

Note that since the parameter s is equal to the arclength at $s = 0$ and the curvature of the body and the vorticity flux are invariant under Euclidian motions [8], substituting the above result in the equation for the heat potential (Eq. (16)) we find that

$$\mu(s) \approx -2F(s)(1-\kappa(s)\sqrt{\pi\nu\,\delta t})^{-1}. \qquad (20)$$

Note that for the case of a cylinder of radius R, the curvature is constant ($\kappa = 1/R$) and for the case of a flat plate the curvature is zero so that the surface potential is only a function of the vorticity flux.

Combining now Eq. (12), Eq. (20), and Eq. (18) we obtain an algorithm for updating the particle strengths in the domain so that the no-slip boundary condition is enforced,

$$\Gamma_j^{n+1} = \Gamma_j^n + h_j^2 \sum_{i=1}^{M} \frac{\gamma_i}{(1-\kappa_i\sqrt{\pi\nu\,\delta t})} \phi(x_{ij}, y_{ij}, \delta t'), \qquad (21)$$

where the i-index refers to the panels, the j-index to the particles, and the n-index to time.

In the present calculations we considered the cases of $\delta t' = \delta t$ (Method 1) and $\delta t' = \delta t/2$ (Method 2) and compare the results of the two approaches. In our computations ϕ is efficiently calculated using tabulated values, thus avoiding the costly evaluation of the error functions that are involved. Moreover, the local character of ϕ requires the interaction of the each panel only with its nearby particles, resulting in a computational cost that scales as $\mathscr{O}(M)$.

3.5. *Test Case I—A Cylinder in Pure Rotation*

We test our numerical scheme by simulating the flow induced by a cylinder (with diameter D) oscillating about

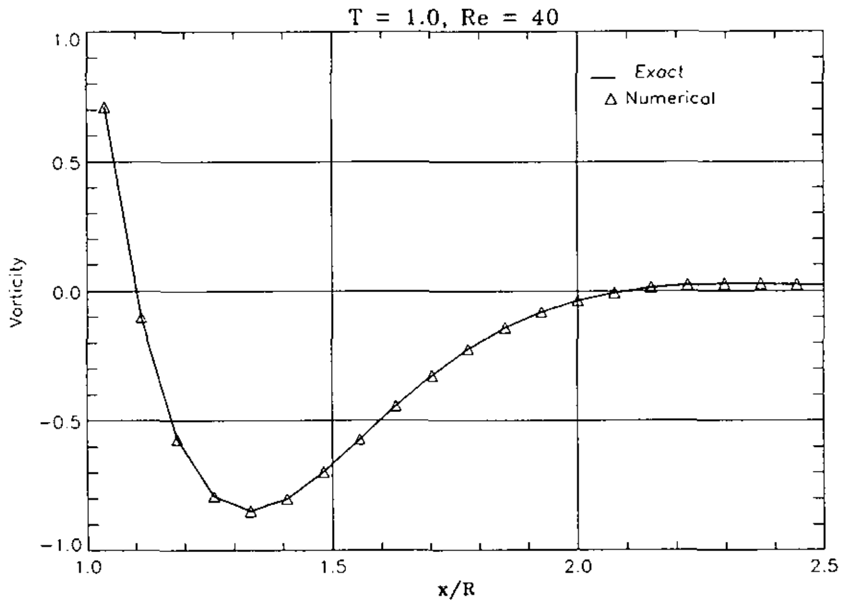

FIG. 3. Vorticity field around a purely rotating cylinder with $Q = 1$, $\sigma = 1$ at $\text{Re} = UD/\nu = 40$ using Method 1.

its axis with angular velocity $Q \sin(\sigma t)$. An analytic solution to this problem may be constructed by assuming the stream-function of the flow to be of the form [7],

$$\Psi = \psi(r)\, e^{i\sigma t}.$$

Under the assumption of symmetry the vorticity equation reduces to a diffusion-type equation whose analytic solution is given by

$$\omega(r, t) = A \cos(\sigma t)[kei_1(c)\, K^-(cr) - ker_1(c)\, K^+(cr)] - A \sin(\sigma t)[kei_1(c)\, K^+(cr) - ker_1(c)\, K^-(cr)]$$

with the definitions

$$K^{\pm}(cr) = ker(cr) \pm kei(cr),$$

$$c = \sqrt{\sigma/v}, \qquad A = \frac{Qc}{\sqrt{2}} \frac{1}{ker_1^2(c) + kei_1^2(c)},$$

where $ker_1(x)$, $kei_1(x)$ and $ker(x)$, $kei(x)$ are the Kelvin's functions of order 1 and 0, respectively [7]. In Fig. 3 we show the results of the computed and analytical vorticity field for $T = 1$ and $v = 0.5$. In order to avoid the computation of the transient solution the vorticity field was initialized with the analytical solution at the end of a period.

4. A FRACTIONAL STEP ALGORITHM

The present vortex method is implemented in a time-stepping algorithm that proceeds by generating the particle trajectories and appropriately modifying the particle strengths. In the present formulation Eqs. (5) are not integrated simultaneously in time but instead a fractional step algorithm is employed. The governing equations are solved via a splitting scheme that accommodates the enforcement of the no-slip boundary condition.

Let us assume that at the nth time step (corresponding to time $t - \delta t$) the vorticity field has been computed (respecting the no-slip boundary condition) and we seek to advance the solution to the next time step (time t). The following two step procedure is implemented:

- *Step* 1. Using as initial conditions $f(\mathbf{x}) = \omega^n(\mathbf{x}^n, n\,\delta t)$ we solve

$$\begin{aligned} &\omega_t + \mathbf{u} \cdot \nabla\omega = v\, \nabla^2 \omega \quad &&\text{in } \mathscr{D} \times [t - \delta t, t] \\ &\omega(\mathbf{x}, t - \delta t) = f(\mathbf{x}) &&\text{in } \mathscr{D}. \end{aligned} \tag{22a}$$

Particles are advanced via the Biot–Savart law and their strength is modified based on the scheme of PSE. Note that no boundary condition is explicitly enforced in this substep. The no-slip condition is enforced in the following stage. Algorithmically then Step 1 may be expressed as:

$$\frac{d\mathbf{x}}{dt} = u^n(\mathbf{x}^n, n\,\delta t)$$

$$\frac{d\omega_1'}{dt} = v\, \nabla^2 \omega_1'.$$

At the end of Step 1 a vorticity field ω_1' has been established in the fluid.

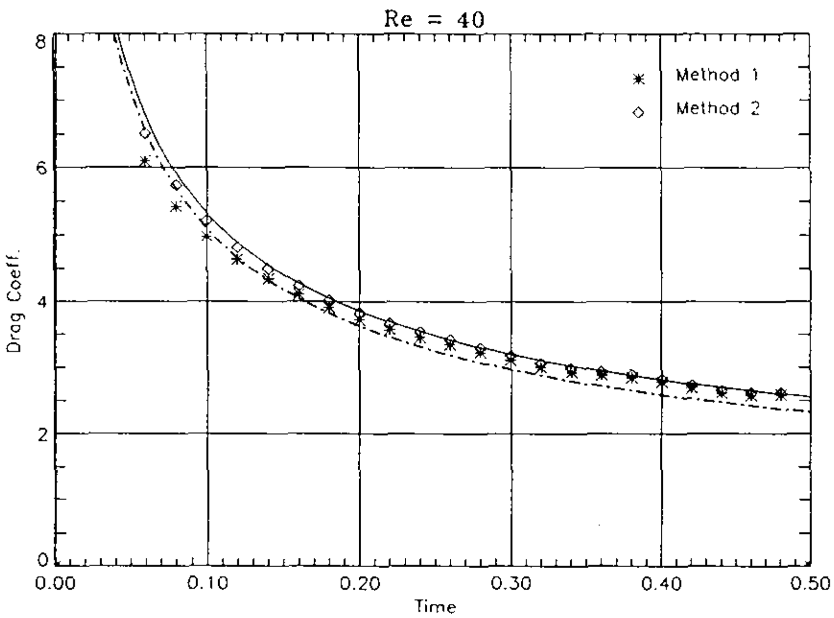

FIG. 4. Early time history of the drag coefficient for an impulsively started circular cylinder for Re = 40 using the two methods for the variation of the heat potential. Solid line [2], dashed line [5], symbols (present computations).

• *Step* 2. The no-slip boundary condition is enforced in this stage by a vorticity (not particle) creation algorithm. The spurious vortex sheet (γ) that is observed on the surface of the body at the end of Step 1 is now computed and then may be translated to a vorticity flux:

$$\gamma \rightarrow \partial\omega/\partial n \qquad \text{on } \partial\mathscr{D}.$$

The computed vorticity flux generates vorticity in the fluid. The vorticity field is augmented by this viscous mechanism as described by:

$$\begin{aligned} &\frac{\partial\omega_2'}{\partial t} - \nu\,\nabla^2\omega_2' = 0 && \text{in } \mathscr{D}\times[t-\delta t, t] \\ &\omega_2'(\mathbf{x}, t-\delta t) = 0 && \text{in } \mathscr{D} \\ &\frac{\partial\omega_2'}{\partial n} = F(\gamma(\omega_1')) && \text{on } \partial\mathscr{D}. \end{aligned} \tag{23}$$

Note that the diffusion equation is solved here with homogeneous initial conditions as the initial vorticity field was taken into account in the previous substep. The solution at Step 2 is a voriticity field ω_2' which we superimpose onto the solution of Step 1 to obtain the vorticity distribution at the next time step

$$\omega^{n+1} = \omega_1' + \omega_2'.$$

4.1. *Test Case II—The Impulsively Started Cylinder*

We examine further the validity of our scheme by simulating the flow around an impulsively started cylinder. The fractional step algorithm described in the previous section is used in conjunction with a fast vortex method to obtain high resolution simulations [12].

To demonstrate the validity of our approach we consider only the early stages of the flow for which analytical solutions exist [2, 5]. At first we compute the drag coefficient

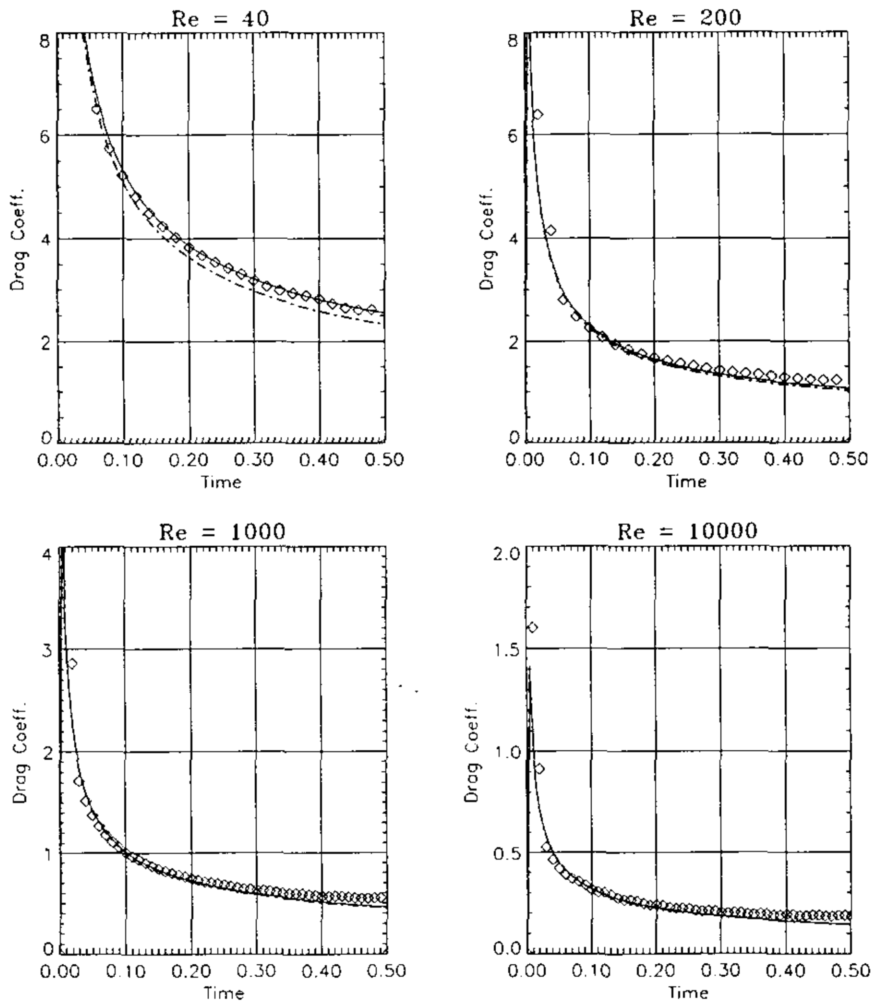

FIG. 5. Linear plot of the early time history of the drag coefficient for an impulsively started circular cylinder. Solid line [2], dashed line [5], symbols (present computations).

using the two methods for the time dependence of the heat potential. In Fig. 4 we observe that Method 2 yields a more accurate solution. Method 1 is equivalent to a first-order (Euler-type) numerical integration of Eq. (17) while Method 2 may be viewed as a second-order (midpoint-type) scheme. In Fig. 5 the drag coefficient as computed by Method 2 is presented and is compared with the analytical solutions for a variety of Reynolds numbers ($\mathrm{Re} = UD/\nu$). One may observe the agreement of the results of the present method with those obtained from the analytical solutions, thus verifying the accuracy of the present approach.

In our computations we used $M \sim \sqrt{(\mathrm{Re}/(UD\,\delta t)}$ with $\delta t = 0.02$ for $\mathrm{Re} = 40$ and 200 and $\delta t = 0.015$ and 0.01 for $\mathrm{Re} = 10^3$ and 10^4, respectively.

5. CONCLUSIONS

We have presented a method for the enforcement of the no-slip boundary condition in the context of the vorticity-velocity formulation of the Navier–Stokes equations. The analysis was for two dimensions but it is easily extended to three dimensions as well. The present scheme is applicable regardless of the numerical method used to discretize these equations. It is especially well suited for vortex methods as it does not require the evaluation of the vorticity and its spatial derivatives at the wall.

The enforcement of the no-slip boundary condition is modelled by a vorticity generation mechanism based on the vorticity flux on the surface of the body. Vorticity enters the fluid by modifying the strength of the existing particles without generating new ones. The present scheme is rigorous and free of ad hoc numerical parameters. It may be combined with the scheme of PSE so that, in vortex methods, all viscous effects can be represented by appropriately modifying the strength of the particles.

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